BARYON STAR MODELS*

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ABSTRACT

We consider, in the framework of general relativity, large masses consisting of identical baryons (whose mass is taken to be that of the neutron). Four assumptions are made as to their interactions, leading to four equations of state, and for each the masses and radii of a number of configurations are calculated for various values of the central number density. The relations of the mass to a parameter t_0 $(=4\sinh^{-1}[\hbar/m_B c(3\pi^2 n_0)^{1/3}]$, where n_0 is the central number density) for each of the four equations of state are strikingly similar.

I. INTRODUCTION

Several discussions have been given (e.g., Oppenheimer and Volkoff 1939; Cameron 1959; Ambartsumyan and Saakyan 1960, 1961; Saakyan 1963) of the problem of the structure of stellar masses composed of elementary particles at ultra-high densities ($\gtrsim$ nuclear density). Such stars are called neutron stars (and are sometimes called hyperon stars). It seems at least possible that such stars exist in nature, presumably as the relics of supernovae. Recent X-ray experiments carried out by means of sounding rockets (Gursky, Giacconi, Paolini, and Rossi 1963; Bowyer, Byram, Chubb, and Friedman 1964) indicate that such stars might even exist.

In this connection it seemed worthwhile to consider whether certain simple alternatives for the equation of state of the particles could be justified on theoretical grounds, and, if so, whether the results would have any similarity to those for a perfect Fermi gas with no interaction.

The first alternative is (potential energy density) $\sim$ (number density)$^{5/3}$. This originated in a hypothesis of Zel'dovich (1959) of a common fermion core "inside" all baryons, which then leads to a strong repulsion (potential $\propto 1/r^2$) when the separation is small and the relative angular momentum is zero. The second was (potential energy density) $\sim$ (number density)^2. This originated in a theory of Zel'dovich (1961) in which baryons interact via a vector meson with a mass.

II. THE EQUATIONS OF HYDROSTATIC EQUILIBRIUM

It has been shown that the relativistic equations of hydrostatic equilibrium are

$$\frac{dM_r}{dr} = 4\pi \frac{1}{c^2} \epsilon r^2 , \tag{1}$$

$$\frac{dP}{dr} = -\frac{G(\epsilon/c^2 + P/c^2)(M_r + 4\pi r^3 P/c^2)}{r(r - 2GM_r/c^2)} . \tag{2}$$

Here M_r is a variable that at the boundary takes on the value of the mass as seen by a distant observer; P the pressure; ϵ the (proper) energy density, including the rest energy; c the velocity of light; G Newton's constant of gravitation; and r the radial Schwarzschild coordinate. The equation of state is determined from the microscopic properties of the medium. In the non-relativistic limit $\epsilon \to \rho c^2$ and $P \ll \rho c^2$, $r \gg GM/c^2$, then equations (1) and (2) reduce to the familiar non-relativistic equations of hydrostatic equilibrium. We now introduce the units (from Oppenheimer and Volkoff 1939)

$$c = G = \frac{1}{8\pi} \frac{m_B c^2}{(\hbar/m_B c)^3} . \tag{3}$$

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In these units the unit of mass is $9.29\, M_\odot$, the unit of length is 13.69 km, and the unit of energy density is $\frac{1}{6}\times$ (rest energy of a baryon)/(volume of a sphere of radius equal to Compton wavelength of baryon). We take m_B as the neutron mass throughout. Equations (1) and (2) become then the equations of Oppenheimer and Volkoff (1939):

$$\frac{du}{dr}=4\pi\epsilon r^2\,,\qquad \frac{dt}{dr}=-4\pi r\left(\frac{dP}{dt}\right)^{-1}\frac{P+\epsilon}{1-2u/r}\,[P+u/(4\pi r^3)]\,, \tag{4}$$

with initial conditions

$$u=0\,,\qquad t=t_0\,,\qquad \text{at}\qquad r=0\,, \tag{5}$$

corresponding to $M_r=0$, $P=P_c$ in the non-relativistic case. Solutions with $u<0$ at the center cannot occur (see Oppenheimer and Volkoff 1939). Here r is the radial Schwarzschild coordinate; u a parameter which at the boundary takes on the value of the observable mass; and t is related to the (proper) particle density by

$$n=(3\pi^2)^{-1}(\hbar/m_B c)^{-3}\sinh^3(t/4)\,. \tag{6}$$

The representation of n was first used in stellar calculations by Chandrasekhar; m_B is the rest mass of one of the (identical) fermions[1] composing the star; ϵ and P are energy density and pressure, known functions[2] of t. We write

$$\epsilon=\epsilon_T+\epsilon_v\,;\qquad P=P_T+P_v\,, \tag{7}$$

where ϵ_T is the kinetic energy density including rest energy density and ϵ_v the potential energy density, and similarly for P.

The units are those of Oppenheimer and Volkoff (1939).

III. EQUATIONS OF STATE

In all cases

$$\epsilon_T=\frac{1}{4\pi}\,(\sinh t-t)\,, \tag{8}$$

$$P_T=\frac{1}{12\pi}\left(\sinh t-8\sinh\frac{t}{2}+3t\right). \tag{9}$$

The expressions for ϵ_T and P_T are the same as those given by Landau and Lifshitz (1958) as one sees from equation (2). They are not those given by Chandrasekhar (1957), since the rest-energy density is included, as it must be; at nuclear densities omission of it would make the parametric form of the equation of state inaccurate (cf. Saakyan 1964).[3]

a) Oppenheimer and Volkoff (1939) give

$$\epsilon_v=P_v=0\,. \tag{10}$$

This is simply the case of neutrons without interactions.

b) Cameron (1959), as modified by Saakyan (1963), gives

$$\begin{aligned}\epsilon_v&=\frac{1}{4\pi}\left(23.9\sinh^8\frac{t}{4}-10.1\sinh^6\frac{t}{4}\right),\\ P_v&=\frac{1}{4\pi}\left(39.9\sinh^8\frac{t}{4}-10.1\sinh^6\frac{t}{4}\right).\end{aligned} \tag{11}$$

[1] We take the mass of the (identical) fermions as being equal to the neutron mass m_B.

[2] These known functions taken together constitute the equation of state.

[3] The non-relativistic limit, as one sees by expanding each expression in powers of t, is

$$P_T=\tfrac{2}{3}(\epsilon_T-m_B c^2 n)\,.$$

This is derived from a nuclear potential given by Skyrme (1959), which is based on the many-body theory of nuclear matter. A three-body effective potential is constructed from which the potential energy density is derived. The predictions of the potential agree well with the data from scattering experiments.

c) Zel'dovich (1959) discussed the hypothesis that the nuclear repulsion ("hard core") is due to a common fermion "inside" all baryons. This led to the conclusion that for different or identical baryons in S states there should appear a strong repulsion at small distances with a potential $\sim \hbar^2/(2\mu R^2)$ (μ the mass, R the separation).

We write for the two-body potential

$$V(r)=\frac{\hbar^2}{m_B}\eta(b-r)\frac{1}{r^2}, \tag{12}$$

where η is the step function and b is a number like the range. Assuming $n^{1/3} \gtrsim b^{-1}$, we get for the potential energy of one particle

$$E_{v\ 1}=2\pi\frac{\hbar^2}{m_B}bn. \tag{13}$$

We have not included the restriction to S states. In an attempt to do so, we say the average wavenumber, k_{av}, satisfies

$$k_{\text{av}}b=\alpha<1. \tag{14}$$

We know that the Fermi wavenumber satisfies

$$\frac{k_{\text{av}}}{k_{\text{F}}}=\beta<1. \tag{15}$$

Then

$$b=\frac{\alpha}{\beta}k_{\text{F}}^{-1}, \tag{16}$$

or

$$b=(3\pi^2)^{-1/3}\frac{\alpha}{\beta}n^{-1/3}<n^{-1/3}, \tag{17}$$

which contradicts our previous assumption. We know, however, that this calculation will give the correct dependence of ϵ_v on n and may hope that the coefficient will not be too far off. Combining equations (17) and (13), we get

$$E_{v,\,1}=(8\pi/3)^{1/3}\frac{\alpha}{\beta}\frac{\hbar^2}{m_B}n^{2/3} \tag{18}$$

for the energy of one particle.

Recalling $\epsilon_v = nE_{v,\,1}$, we obtain, in our units, just

$$\epsilon_v=\frac{16}{9\pi^2}\frac{\alpha}{\beta}\sinh^5\frac{t}{4}. \tag{19}$$

To get P_v, use

$$P_v=\sinh^6\frac{t}{4}\frac{\partial}{\partial[\sinh^3(t/4)]}\left[\frac{\epsilon_v}{\sinh^3(t/4)}\right] \tag{20}$$

to get

$$P_v=\frac{32}{27\pi^2}\frac{\alpha}{\beta}\sinh^5\frac{t}{4}. \tag{21}$$

d) Zel'dovich (1961) also considered the question of the most rigid equation of state compatible with the theory of relativity. He showed that baryons interacting via a vector field mediated by massive quanta resulted in (assuming $n^{-1/3} < 1/\mu$)

$$E_{v\ 1}=2\pi g^2n^2/\mu^2, \tag{22}$$

where $m = \hbar\mu$ is the mass of the meson and g is the baryonic charge of the baryons. (We take $c = 1$.) Again using $\epsilon_v = nE_{v,1}$ and equation (20), and defining

$$\gamma = \frac{g^2/\hbar}{(m/m_B)^2}, \tag{23}$$

we get (in our units) just

$$\epsilon_v = P_v = \frac{16}{9\pi^2}\gamma \sinh^6 \frac{t}{4}. \tag{24}$$

Zel'dovich also assumed, and it is material to his argument, that

$$1 < \gamma < \left(\frac{m_B}{m}\right)^3. \tag{25}$$

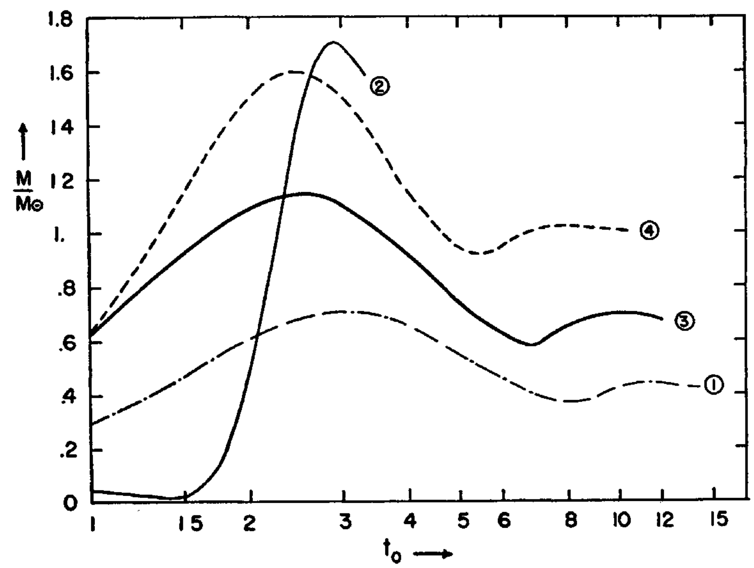

FIG. 1.—Curve *1*, first equation of state (eq. [10]); curve *2*, second equation of state (eq. [11]); curve *3*, third equation of state (eqs. [19]–[21]) ($\alpha/\beta = 1$); curve *4*, fourth equation of state (eq. [24]) ($\gamma = 3$).

He has taken, too, $\epsilon_T = m_B n$, but he states clearly that this is an approximation; for exactness we must use, and do use, equation (8).

Note that for this equation of state the speed of sound is equal to that of light.

IV. RELATIVISTIC RESTRICTIONS ON THE EQUATION OF STATE

The equation of state must not lead to a prediction which contradicts the theory of relativity. In particular, for very large n, $P \propto n^s$, with $s > 2$, is not permitted; otherwise $v_{\text{sound}} > c$. Within this framework it is arguable that the equations given in (*c*) and (*d*) above are quite simple and have some justification from elementary particle theory. Their simplicity, and the radical differences in the dependence of P on n between them, make the unusually similar results for (*a*), (*c*), and (*d*) (see Zel'dovich 1961) interesting.

V. RESULTS

a) Equations (4) were integrated numerically for 15–20 values of t_0, for each equation of state, on an IBM 7094 computer. The initial conditions (eq. [5]) define the starting values, except in the case of infinite central density (see below); the integration is stopped when $t = 10^{-3}\, t_0$. The values of u and r at this point are M and R.

Note that we take $\alpha/\beta = 1$ and $\gamma = 3$, respectively, and that no values of t_0 which would make $P(t_0) > \epsilon(t_0)$ are used in the second equation of state.

The results are given in Figure 1 and in Table 1. The last value is also that for infinite density, except for the second equation of state, in which it is the last solution for which $P(t_0) < \epsilon(t_0)$.

TABLE 1

A. First Equation of State, Equation (10)

t_0	M	M_N	R
1 0	0 3015	0 3043	20 76
1 5	4793	488	16 14
2 0	6121	6288	13 13
2 3	6643	6855	11 74
2 6	6964	7209	10 57
2 8	7077	7336	9 877
3 0	7120	7384	9 270
3 3	7073	7330	8 455
3 6	6920	7153	7 754
4 0	6602	6780	6 970
5 0	5561	5546	5 627
6 0	4574	4381	5 000
7 0	3902	3603	5 035
7 5	3731	3410	5 298
8 0	3688	3362	5 679
8 5	3764	3445	6 085
9 0	3921	3619	6 401
10 0	4259	3993	6 597
11 0	4416	4168	6 431
12 0	4399	4150	6 221
13 0	4315	4055	6 104
14 0	4246	3977	6 093
15 0	4222	3950	6 140
20 0	4267	4001	6 182
30.0.	4262	0 3995	6 186
Infinite	0 4262		6 185

B. Second Equation of State, Equation (11)

t_0	M	M_N	R
0 5	0 04571	0 04575	27 14
1 0	0 04489	0 04493	22 09
1 3	0 03707	0 03709	22 41
1 5	0 03339	0 03342	8 518
1 6	0 06479	0 06513	6 904
1 8	0 2093	0 2140	7 147
2 0	0 4955	0 5220	8 010
2 2	0 8931	0 9797	8 633
2 4	1 286	1 473	8 821
2 5	1 442	1 685	8 765
2 6	1 560	1 854	8 624
2 7	1 640	1 974	8 428
2 8	1 685	2 044	8 194
2 9	1 701	2 070	7 940
3 0	1 694	2 058	7 681
3 2	1 636	1 958	7 184
3 3	1 594	1 886	6 960

TABLE 1—*Continued*

C. Third Equation of State, Equations (19)–(21)
($\alpha/\beta = 1$)

t_0	M	M_N	R
1 0	0 6292	0 6385	26 53
1 5	0 9325	0 9582	20 17
2 0	1 100	1 142	16 03
2 2	1 131	1 176	14 73
2 4	1 143	1 190	13 60
2 5	1 144	1 191	13 08
2 6	1 141	1 188	12 60
2 8	1 126	1 171	11 71
3 0	1 103	1 143	10 93
4 0	0 9165	0 9170	8 237
5 0	0 7297	0 6905	7 064
6 0	0 6124	0 5520	7 194
6 5	0 5906	0 5270	7 717
7 0	0 5957	0 5327	8 393
7 5	0 6212	0 5614	8 970
8 0	0 6539	0 5980	9 272
9 0	0 6969	0 6465	9 186
9 5	0 7019	0 6522	9 009
10 0	0 6993	0 6492	8 844
11 0	0 6853	0 6332	8 660
12 0	0 6754	0 6219	8 665
15 0	0 6796	0 6267	8 781
20 0	0 6788	0 6258	8 762
40.0.	0 6788	0 6259	8 762
Infinite	0 6788		8 760

D. Fourth Equation of State, Equation (24)
($\gamma = 3$)

t_0	M	M_N	R
1 0	0 6232	0 6341	24 18
1 3	0 9459	0 9749	21 02
1 5	1 148	1 194	19 25
1 7	1 320	1 387	17 64
2 0	1 503	1 598	15 50
2 2	1 569	1 677	14 23
2 3	1 587	1 699	13 64
2 4	1 595	1 709	13 08
2 5	1 595	1 709	12 55
2 7	1 573	1 681	11 58
3 0	1 501	1 586	10 35
3 2	1 436	1 501	9 659
3 5	1 331	1 359	8 824
3 7	1 259	1 264	8 390
4 0	1 159	1 130	7 914
4 5	1 024	0 9531	7 568
5 0	0 9440	0 8512	7 733
5 3	0 9252	0 8279	8 017
5 5	0 9242	0 8266	8 242
5 7	0 9307	0 8345	8 467
6 0	0 9502	0 8582	8 755
7 0	1 018	0 9405	9 037
8 0	1 027	0 9521	8 845
9 0	1 015	0 9368	8 747
10 0	1 010	0 9305	8 765
12 0	1 013	0 9342	8 794
30.0.	1 013	0 9338	8 788
Infinite	1 013		8 787

b) It is possible, as Oppenheimer and Volkoff showed, to integrate the equations even if the central density is infinite. We obtain in the several cases the solutions

(i) $$\frac{u}{r}=\frac{3}{14}, \qquad \frac{1}{r^2}=\frac{7}{3}e^{t}, \tag{26}$$

(ii) . . . .

(see above; $P[t_0] > \epsilon[t_0]$ is not permitted).

(iii) $$\frac{u}{r}=\frac{12}{49}, \qquad \frac{1}{r^2}=\frac{49}{54\pi}\frac{\alpha}{\beta}e^{5t/4}, \tag{27}$$

(iv) $$\frac{u}{r}=\frac{1}{4}, \qquad \frac{1}{r^2}=\frac{4}{9\pi}\gamma e^{3t/2}. \tag{28}$$

These are taken in all cases as being accurate out to values of r such that $t = 30$, and the integration is done numerically from that point.

VI. DISCUSSION

The general character of the M versus t_0 curves (Fig. 1) is the same for the first, third, and fourth equations of state. We have a gradual rise to an absolute maximum around $t_0 = 2.5$–3, a decline to a relative minimum between 5.5 and 8.0, a rise to a relative maximum between 7.5 and 11, and a decline to the infinite-density-solution value, which is attained (to three significant figures) between 10 and 20.

The behavior of R as a function of t_0 for each equation of state is quite similar to the curves *1a* and *2a* of Figure 2 of the paper by Arbartsumyan and Saakyan (1961).

We see that, except for the second equation of state (which for large t is unphysical), the M versus t_0 curves are quite similar. The features present in the solution curve for the simplest possible case (Oppenheimer and Volkoff) are present in the others also. In particular, there is an absolute maximum and a second relative maximum. One might think that this would indicate the presence of a second stability region from the first minimum to the second maximum: It seems reasonable that, wherever the slope of the M versus t_0 curve is positive, the configuration should be stable. Of course, questions of stability can be decided only by detailed calculation. However, Misner and Zapolsky (1964), following an earlier work by Chandrasekhar (1964), have shown that the equilibrium, for polytropes, is unstable in this range of values of the density.

Of course, the existence of an absolute maximum for a given equation of state has the consequences pointed out by Oppenheimer and Volkoff (1939). No stable configuration is possible for $M > M_{\max}$. A star having a greater mass must lose part of it by some mechanism or collapse toward the Schwarzschild singularity.

VII. COMPARISON WITH PREVIOUS WORK

a) *First Equation of State*

The curve of M versus t_0 does not agree with any published works (e.g., Oppenheimer and Volkoff 1939; Saakyan 1963), but Saakyan's (1962) comments seem to indicate that he is aware that two relative maxima must exist in this case also. Slight differences were found between the masses and radii of Oppenheimer and Volkoff's (1939) and ours. Our value for the mass for $t_0 = \infty$ does not agree with Oppenheimer and Volkoff's (1939) but seems to be very close to the value given in Saakyan (1963). In any event a change of stabilities occurs long before the ultra-density regime, at the first mass peak.

b) Second Equation of State

We confirm the results of Saakyan (1963) but numerical comparison could not be made, since Saakyan gave no table. Certain discrepancies exist between Cameron's (1959) results and ours, including one (for $\rho_c = 3 \times 10^{14}$ c.g.s.) in a region where Saakyan says his and Cameron's results were identical. Note that Saakyan does not include one of Cameron's points in his graph, viz., $\rho_c = 2 \times 10^{14}$ c.g.s. ($t_0 = 1.258$).

Table 1 gives R, M, and M_N for each model and for each value of t_0.

VIII. BINDING ENERGIES

We have also calculated the quantity

$$u_N(R) = m_B c^2 \int_0^R 4\pi r^2 \left(1 - 2\frac{u}{r}\right)^{-1/2} n\, dr\,, \tag{29}$$

where n is given by equation (6) for all the models except the infinite-density ones. The column M_N in Table 1 gives the values of $u_N(R)$ in solar-mass units.

M_N is simply the total rest mass of the baryons composing the star. If $M_N - M$ is positive, we have binding, and conversely. The table shows that, roughly speaking, for t_0 somewhat past the first maximum, the configurations are unbound.

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